# Appendix B
# Special Coordinate Systems

This appendix lists basic relations for some special orthogonal curvilinear coordinate systems. Coordinates and base vectors are expressed in terms of the Cartesian system $(x, y, z)$. In addition, physical stress and deformation components are used.

## B.1 Cylindrical Polar Coordinates

The geometry in shown in Fig. B.1.

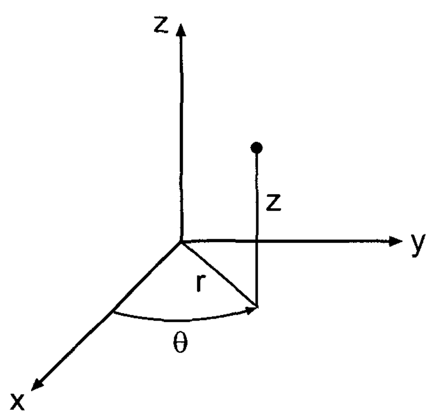


Fig. B.1 Cylindrical polar coordinate system.

## Geometry:

$$(x^1, x^2, x^3) = (r, \theta, z)$$

$$\begin{aligned} x &= r\cos\theta, \qquad r = (x^2 + y^2)^{1/2} \\ y &= r\sin\theta, \qquad \theta = \tan^{-1}(y/x) \end{aligned}$$

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta = \mathbf{e}_r \\ \mathbf{g}_2 &= r(-\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta) = r\mathbf{e}_\theta \\ \mathbf{g}_3 &= \mathbf{e}_z \end{aligned}$$

$$\begin{aligned} g_{11} &= 1, \qquad g_{22} = r^2, \qquad g_{33} = 1 \\ g^{11} &= 1, \qquad g^{22} = r^{-2}, \qquad g^{33} = 1 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{22} &= -r \\ \Gamma^2_{12} &= \Gamma^2_{21} = r^{-1} \\ \text{Other } \Gamma^k_{ij} &= 0 \end{aligned}$$

## Differential operators:

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

## Equations of motion:

$$\begin{aligned} \frac{\partial \hat{\sigma}^{rr}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta r}}{\partial \theta} + \frac{\partial \hat{\sigma}^{zr}}{\partial z} + \frac{1}{r}(\hat{\sigma}^{rr} - \hat{\sigma}^{\theta\theta}) + \hat{f}^r &= \rho \hat{a}^r \\ \frac{\partial \hat{\sigma}^{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta\theta}}{\partial \theta} + \frac{\partial \hat{\sigma}^{z\theta}}{\partial z} + \frac{2\hat{\sigma}^{r\theta}}{r} + \hat{f}^\theta &= \rho \hat{a}^\theta \\ \frac{\partial \hat{\sigma}^{rz}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta z}}{\partial \theta} + \frac{\partial \hat{\sigma}^{zz}}{\partial z} + \frac{\hat{\sigma}^{rz}}{r} + \hat{f}^z &= \rho \hat{a}^z \end{aligned}$$

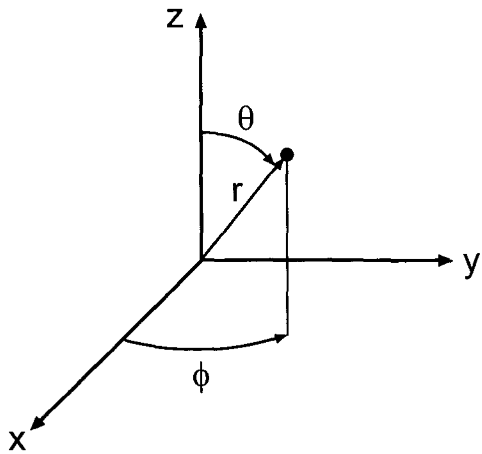


Fig. B.2 Spherical polar coordinate system.

**Deformation gradient:**

$$\mathbf{F}_{(\mathbf{e}_i\mathbf{e}_j)} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix}$$

## B.2 Spherical Polar Coordinates

The geometry in shown in Fig. B.2.

**Geometry:**

$$(x^1, x^2, x^3) = (r, \theta, \phi)$$

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g}_1 &= \mathbf{e}_x\sin\theta\cos\phi + \mathbf{e}_y\sin\theta\sin\phi + \mathbf{e}_z\cos\theta = \mathbf{e}_r \\
\mathbf{g}_2 &= r(\mathbf{e}_x\cos\theta\cos\phi + \mathbf{e}_y\cos\theta\sin\phi - \mathbf{e}_z\sin\theta) = r\,\mathbf{e}_\theta \\
\mathbf{g}_3 &= r\sin\theta(-\mathbf{e}_x\sin\phi + \mathbf{e}_y\cos\phi) = r\sin\theta\,\mathbf{e}_\phi
\end{aligned}
$$

$$
\begin{aligned}
g_{11} &= 1, \qquad g_{22} = r^2, \qquad g_{33} = (r\sin\theta)^2 \\
g^{11} &= 1, \qquad g^{22} = r^{-2}, \qquad g^{33} = (r\sin\theta)^{-2}
\end{aligned}
$$

$$
\begin{array}{ll}
\Gamma^1_{22} = -r, & \Gamma^1_{33} = -r\sin^2\theta \\
\Gamma^2_{12} = \Gamma^2_{21} = r^{-1} & \Gamma^2_{33} = -\sin\theta\cos\theta \\
\Gamma^3_{13} = \Gamma^3_{31} = r^{-1} & \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \\
\text{Other } \Gamma^k_{ij} = 0 &
\end{array}
$$

**Differential operators:**

$$
\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}
$$

$$
\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}
$$

**Equations of motion:**

$$
\begin{aligned}
\frac{\partial \hat{\sigma}^{rr}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \hat{\sigma}^{\phi r}}{\partial \phi} & \\
+\frac{1}{r}\left(2\hat{\sigma}^{rr} - \hat{\sigma}^{\theta\theta} - \hat{\sigma}^{\phi\phi} + \hat{\sigma}^{\theta r}\cot\phi\right) + \hat{f}^{r} &= \rho\hat{a}^{r} \\
\frac{\partial \hat{\sigma}^{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \hat{\sigma}^{\phi\theta}}{\partial \phi} & \\
+\frac{1}{r}\left[2\hat{\sigma}^{r\theta} + \hat{\sigma}^{\theta r} + (\hat{\sigma}^{\theta\theta} - \hat{\sigma}^{\phi\phi})\cot\theta\right] + \hat{f}^{\theta} &= \rho\hat{a}^{\theta} \\
\frac{\partial \hat{\sigma}^{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}^{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \hat{\sigma}^{\phi\phi}}{\partial \phi} & \\
+\frac{1}{r}(2\hat{\sigma}^{r\phi} + \hat{\sigma}^{\phi r} + 2\hat{\sigma}^{\theta\phi}\cot\theta + \hat{f}^{\phi} &= \rho\hat{a}^{\phi}
\end{aligned}
$$

**Deformation gradient:**

$$
\mathbf{F}_{(\mathbf{e}_i\mathbf{e}_j)} = \begin{bmatrix}
\dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{1}{R\sin\Theta}\dfrac{\partial r}{\partial \Phi} \\
r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & \dfrac{r}{R\sin\Theta}\dfrac{\partial \theta}{\partial \Phi} \\
r\sin\theta\dfrac{\partial \phi}{\partial R} & \dfrac{r\sin\theta}{R}\dfrac{\partial \phi}{\partial \Theta} & \dfrac{r\sin\theta}{R\sin\Theta}\dfrac{\partial \phi}{\partial \Theta}
\end{bmatrix}
$$

## B.3 Toroidal Coordinates

Because of the complexity of some general expressions, only basic geometric relations are listed below (see Fig. B.3).

$$(x^1, x^2, x^3) = (r, \theta, \phi)$$

$$
\begin{aligned}
x &= (b + r\cos\phi)\cos\theta \\
y &= (b + r\cos\phi)\sin\theta \\
z &= r\sin\phi
\end{aligned}
$$

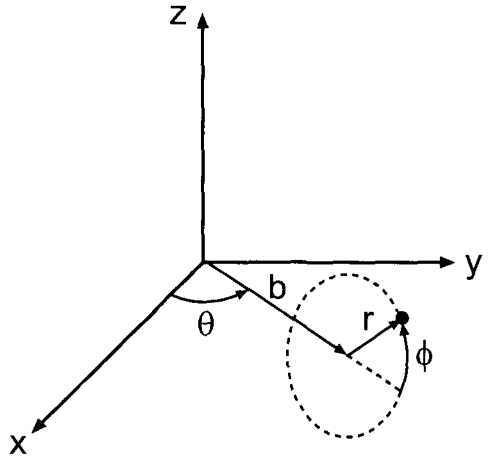


Fig. B.3 Toroidal coordinate system.

$$
\begin{aligned}
\mathbf{g}_1 &= \cos\phi(\mathbf{e}_x\cos\theta + \mathbf{e}_y\sin\theta) + \mathbf{e}_z\sin\phi \\
\mathbf{g}_2 &= (b + r\cos\phi)(-\mathbf{e}_x\sin\theta + \mathbf{e}_y\cos\theta) \\
\mathbf{g}_3 &= -r\sin\phi(\mathbf{e}_x\cos\theta + \mathbf{e}_y\sin\theta) + \mathbf{e}_z r\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
g_{11} &= 1, \qquad g_{22} = (b + r\cos\phi)^2, \qquad g_{33} = r^2 \\
g^{11} &= 1, \qquad g^{22} = (b + r\cos\phi)^{-2}, \qquad g^{33} = r^{-2}
\end{aligned}
$$

$$
\begin{array}{ll}
\Gamma^1_{22} = -(b + r\cos\phi)\cos\phi, & \Gamma^1_{33} = -r \\
\Gamma^2_{12} = \Gamma^2_{21} = (b + r\cos\phi)^{-1}\cos\phi & \Gamma^2_{23} = \Gamma^2_{32} = -(b + r\cos\phi)^{-1} r\sin\phi \\
\Gamma^3_{13} = \Gamma^3_{31} = r^{-1} & \Gamma^3_{22} = (b + r\cos\phi) r^{-1}\sin\phi \\
\text{Other } \Gamma^k_{ij} = 0 &
\end{array}
$$